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ON COCYCLE CONJUGACY OF QUASIFREE ENDOMORPHISMS SEMIGROUPS ON THE CAR ALGEBRA

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W. Arveson has described a cocycle conjugacy class $\mathcal{U}(\alpha)$ of E_0-semigroup α on $\mathcal{B}(\mathcal{H})$ which is a factor of type I. Under some conditions on α there is a E_0-semigroup $\beta \in \mathcal{U}(\alpha)$ being a flow of shifts in the sense of R.T.Powers (see [11]). We study quasifree endomorphisms semigroups α on the hyperfinite factor $\mathcal{M} = \pi(\mathcal{A}(\mathcal{K}))''$ generated by the representation π of the algebra of the canonical anticommutation relations $\mathcal{A}(\mathcal{K})$ over a separable Hilbert space $\mathcal{K}$. The type of $\mathcal{M}$ can be I, II or III depending on π. The cocycle conjugacy class $\mathcal{U}(\alpha)$ is described in the terms of initial isometrical semigroup in $\mathcal{K}$ and an analogue of the Arveson result for the hyperfinite factor $\mathcal{M}$ of type II_1 and $III_\lambda, 0 < \lambda < 1$, is introduced.

1. Introduction. Let $\mathcal{M}$ be the W^*-algebra acting in a Hilbert space. One-parameter unital semigroup $\alpha_t \in End(\mathcal{M}),\ t \geq 0$, is called a E_0-semigroup if every function $\eta(\alpha_t(x))$ is continuous in t for $x \in \mathcal{M}$ and $\eta \in \mathcal{M}_*$. Given a E_0-semigroup α one can define its generator $\delta(a) = lim_{t\to 0}\frac{\alpha_t(a)-a}{t}$ for $a \in dom\delta$, where $dom\delta$ is σ-week dense in $\mathcal{M}$ (see [14]). Two E_0-semigroups α and β are called to be cocycle conjugate if there is a strong continuous family of unitaries $U_t \in \mathcal{M},\ t \geq 0$, named a cocycle, such that $\beta_t(\cdot) = U_t\alpha_t(\cdot)U_t^*,\ U_{t+s} = U_t\alpha_t(U_s),\ t, s \geq 0$ (see [11,14,15]). Notice that if semigroups α and β have generators differ on a bounded derivation, then α and β are cocycle conjugate (see [15]). This case is associated with the differentiable cocycle $(U_t)_{t\geq 0}$ and it is not the general one. The discussion on the cocycle conjugacy of automorphisms semigroups on the W^*-algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ one can see in monograph [14]. A notion of the cocycle conjugacy of endomorphisms semigroups on the W^*-algebra $\mathcal{M}$ was given by W. Arveson. He studied the case of $\mathcal{M} = \mathcal{B}(\mathcal{H})$ in [11].

Let $\mathcal{A} = \mathcal{A}(\mathcal{K})$ be the C^*-algebra of the canonical anticommutation relations (CAR) over a Hilbert space $\mathcal{K}$. It means that there is a map $f \to a(f)$ from $\mathcal{K}$ to the C^*-algebra $\mathcal{A}$ (with the unit $\mathbf{1}$, the involution * and the norm $||\cdot||$) satisfying the following properties:

1) $a(\lambda f + g) = \lambda^* a(f) + a(g)$ for all $f, g \in \mathcal{K},\ \lambda \in \mathbf{C}$,

2) (CAR) $a(f)a(g) + a(g)a(f) = 0$,

$$a^*(f)a(g) + a(g)a^*(f) = (g, f)\mathbf{1},$$

3) the polinomials in all $a(f), a^*(g)$ are dense in $\mathcal{A}$ by the norm, $||a(f)|| = ||f||_{\mathcal{K}}$.

Every state on $\mathcal{A}$ that is a positive linear functional $\phi \in \mathcal{A}^*,\ \phi(\mathbf{1}) = \mathit{1}$, is determined by its values on (Wick) normal ordered monomials $a^*(f_1)...a^*(f_m)a(g_1)...a(g_n)$. The operator $R,\ 0 < R < 1$, in $\mathcal{B}(\mathcal{K})$ determines a state ω_R satisfying the condition

$$\omega_R(a^*(f_m)...a^*(f_1)a(g_1)...a(g_n)) = \delta_{nm} det((f_i, Rg_j)).$$

Such ω_R is called a quasifree state. Let us define a representation π_R of the C^*-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H} = \mathcal{F}(\mathcal{K}) \otimes \mathcal{F}(\mathcal{K})$ by the formula

$$\pi_R(a(f)) = a((1-R)^{1/2} f) \otimes \Gamma + \mathit{1} \otimes a^*(JR^{1/2} f),\ f \in \mathcal{K},$$

$$\pi_R(\mathbf{1}) = \mathrm{Id},$$

where $\mathcal{F}(\mathcal{K})$ is the antisymmetric (fermion) Fock space over K with a vacuum vector Ω, J is some antiunitary in K, Γ is a hermitian unitary operator completely defined by the condition $\Gamma a(f) = -a(f)\Gamma, f \in h, \Gamma\Omega = \Omega$. In this representation the state ω_R becomes the vector one, $\omega_R(x) = (\Omega \otimes \Omega, \pi_R(x)\Omega \otimes \Omega),\ x \in \mathcal{A}$. Therefore the triple $(\pi_R, \mathcal{H}_R, \mathit{\Omega} \otimes \mathit{\Omega})$, where $\mathcal{H}_R = \overline{\pi_R(\mathcal{A})\mathit{\Omega} \otimes \mathit{\Omega}}$, is the Gelfand-Neumark-Segal (GNS) representation of the C^*-algebra $\mathcal{A}$ associated with the state ω_R. The state ω_R is pure if and only if $R = P, P^2 = P$ is an orthogonal projection. In this case the GNS representation π_P acting in a Hilbert space $\mathcal{H} = \mathcal{F}((I-P)\mathcal{K}) \otimes \mathcal{F}(JPJ\mathcal{K})$ yelds the W^*-algebra $\mathcal{M}_P(\mathcal{K}) = \pi_P(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$. Setting a twopoints function of the state ω to be

$$\omega(a^*(f)a(g)) = \nu(f, g),\ f, g \in \mathcal{K},$$

and fixing a numerical parameter $\nu \in [0, 1/2]$ one can consider the quasifree state $\omega = \omega_\nu$ assosiated with the operator $R = \nu\mathbf{1}$ in $\mathcal{K}$. If $\nu \neq 0$, then ω is exact ($\omega(x^*x) = 0$ implies $x = 0$). In the GNS representation π_ν acting in a Hilbert space

$\mathcal{H} = \mathcal{F}(\mathcal{K}) \otimes \mathcal{F}(\mathcal{K})$ the C^*-algebra $\pi_\nu(\mathcal{A})$ generates the W^*-algebra $\mathcal{M}_\nu = \mathcal{M}_\nu(\mathcal{K}) = \pi_\nu(\mathcal{A}(\mathcal{K}))''$. If $0 < \nu < 1/2$, then $\mathcal{M}_\nu$ is a hyperfinite factor of type III_λ, where $\lambda = \nu/(1-\nu)$. In the case of $\nu = 1/2$ the state ω is a trace and $\mathcal{M}_\nu$ is a hyperfinite factor of type II_1. The case of $\nu = 0$ is associated with the vacuum state and the Fock representation with $\mathcal{M}_0 = \mathcal{B}(\mathcal{F}(\mathcal{K}))$ (see [9-10,16-20]).

Let a quasifree endomorphism α on the algebra $\pi_R(\mathcal{A})$ act on the generating elements by the formula $\alpha(\pi_R(a(f))) = \pi_R(a(Vf)),\ f \in \mathcal{K}$, where V is an isometry in a Hilbert space K commuting with R. If $kerR = ker(I-R) = 0$ the state ω_R is exact and the vector $\Omega \otimes \Omega$ is separating for the W^*-factor $\mathcal{M}_R = \pi_R(\mathcal{A})''$. It allowes to show that α can be extended to a quasifree endomorphism of $\mathcal{M}_R$ (see [16]). We denote this endomorpism by $B_R(V)$. The procedure of the passage from an operator in a Hilbert space $\mathcal{K}$ to a map on the algebra $\mathcal{M}_R$ is called the (quasifree) lifting. If $(V_t)_{t\geq 0}$ is a C_0-semigroup of isometries in $\mathcal{K}$ commuting with R, then the semigroup $(B_R(V_t))_{t\geq 0}$ is a E_0-semigroup on the W^*-factor $\mathcal{M}_R$. We call these semigroups the quasifree ones (see [9-10,16-19]).

The endomorphism α of W^*-algebra $\mathcal{M}$ is called a shift if $\cap_{n=1}^{+\infty}\alpha^n(\mathcal{M}) = \mathbf{C1}$. The E_0-semigroup $(\alpha_t)_{t\geq 0}$ is called a flow of shifts if α_t is a shift for every fix $t > 0$. In [12] R.T. Powers introduced a flow of shifts on the W^*-algebra $\mathcal{M}_0 = \mathcal{B}(\mathcal{F}(\mathcal{K}))$. It was obtained by an extension of the semigroup $(\alpha_t)_{t\geq 0}$ acting on the generating elements of the C^*-algebra $\pi_0(\mathcal{A}(\mathcal{K})),\ \mathcal{K} = L_2(0, +\infty)$, by the formula $\alpha_t(\pi_0(a(f))) = \pi_0(a(S_t f)),\ t \geq 0,\ f \in \mathcal{K}$, where $(S_t)_{t\geq 0}$ is a C_0-semigroup of right shifts in $\mathcal{K}$ defined by the formula $(S_t f)(x) = f(x-t)$ for $x > t$ and $(S_t f)(x) = 0$ for $0 < x < t,\ f \in \mathcal{K}$. In [13] it is asserted that the quasifree E_0-semigroup $(B_\nu(S_t))_{t\geq 0}$ on the hyperfinite factor $\mathcal{M}_\nu,\ 0 < \nu \leq 1/2$, consists of shifts.

In [11] W. Arveson posed a question: to describe the class of cocycle conjugacy of the flow of shifts on the W^*-algebra $\mathcal{M}$ defined by R.T. Powers in [12]. One can see the answer on this question in the case of $\mathcal{M} = \mathcal{B}(\mathcal{H})$ in [11]. We investigate cocycle conjugacy of quasifree automorphisms and endomorphisms semigroups on the hyperfinite factors of type II_1 and $III_\lambda,\ 0 < \lambda < 1$.

In what follows we denote the trace class, the Hilbert Schmidt class, compact operators and the Hilbert-Schmidt norm by symbols s_1, s_2, s_∞ and $||\cdot||_2$ correspondently.

2. An extension on $\mathcal{B}(\mathcal{H})$ of quasifree automorpisms of hyperfinite

factors $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$.

Fix a positive operator $R,\ 0 < R < I,\ kerR = ker(I-R) = 0$, and an antiunitary operator J in K and construct a representation π of the algebra $\mathcal{A}(\mathcal{K}\oplus\mathcal{K})$ in a Hilbert space $\mathcal{H} = \mathcal{F}(\mathcal{K}) \otimes \mathcal{F}(\mathcal{K})$ by the formula

$$\pi(a(f \oplus 0)) = a((1-R)^{1/2}f) \otimes \Gamma + 1 \otimes a\ (R^{1/2}Jf),$$

$$\pi(a(0 \oplus f)) = a(R^{1/2}f) \otimes \Gamma - 1 \otimes a\ ((1-R)^{1/2}Jf),$$

$f \in \mathcal{K}$. Here Γ is a unitary operator completely defined by the relations $\Gamma a(f) = -a(f)\Gamma,\ \Gamma\Omega = \Omega,\ f \in \mathcal{K}$. Define the hyperfinite factors $M_R = \pi(\mathcal{A}(\mathcal{K} \oplus 0))''$ and $M_P = \pi(\mathcal{A}(\mathcal{K} \oplus \mathcal{K}))'' = \mathcal{B}(\mathcal{H})$ and consider two vector states on its,

$$\omega_R(x) =< \Omega \otimes \Omega, \pi(x \oplus 0)\Omega \otimes \Omega >,\ x \in \mathcal{A}(\mathcal{K} \oplus 0).$$

$$\omega_P(x) =< \Omega \otimes \Omega, \pi(x)\Omega \otimes \Omega >,\ x \in \mathcal{A}(\mathcal{K} \oplus \mathcal{K}).$$

Note that

$$\omega_R(\pi(a^*(f)a(g))) = (f, Rg),\ f, g \in \mathcal{K},$$

$$\omega_P(\pi(a^*(f)a(g))) = (f, Pg),\ f, g \in \mathcal{K} \oplus \mathcal{K},$$

where $P = \begin{pmatrix} R & R^{1/2}(I-R)^{1/2} \\ R^{1/2}(I-R)^{1/2} & I-R \end{pmatrix}$ is an orthogonal projection in a Hilbert space $\mathcal{K} \oplus \mathcal{K}$. The state ω_R is exact and the state ω_P is pure and obtained by the purification procedure (see [19]) from ω_R. Operators

$$b(f) = \Gamma \otimes \Gamma\pi(a(0 \oplus f)),\ b^*(f) = \pi(a^*(0 \oplus f))\Gamma \otimes \Gamma,\ f \in K,$$

generate the commutant M_R' and satisfy the relation $b(f) = \mathcal{J}\pi(a(f \oplus 0))\mathcal{J},\ f \in K$, where $\mathcal{J}$ is a modular involution on M_R associated with ω_R (see Appendix). Let V and W be isometrical operators in K commuting with R. Consider quasifree endomorphisms α of M_R and β of its commutant M_R' obtained by the lifting of V and W: $\alpha(\pi(a(f \oplus 0))) = \pi(a(Vf \oplus 0)),\ \beta(b(f)) = b(Wf),\ f \in K$. Consider minimal unitary dilations V' and W' of the operators V and W acting in the same Hilbert space K', $K \subset K'$ and a positive contraction R' in K' such that $V'R = R'V,\ W'R = R'W$ (on the existence of R' see in [17]). Determine a quasifree endomorphism θ of the C^*-algebra generated by M_R and M_R' such that $\theta|_{M_R} = \alpha,\ \theta|_{M_R'} = \beta$.

Theorem 1. *The endomorphism θ defined by the quasifree lifting of the isometrical operators V and W, can be extended on M_P if and only if the following inclusion holds, $R'^{1/2}(I-R')^{1/2}(V'-W') \in s_2$.*

Remark. *The condition of the proposition is sufficient for the cocycle conjugacy of endomorphic semigroups on M_R obtained by the quasifree lifting of the isometrical operators V and W included in the semigroups $\mathcal{V}$ and $\mathcal{W}$ in a Hilbert space K (see below).*

Proof.

As it was proved by H.Araki (see [9-10]), any quasifree *-automorphism θ' given on the C^*-algebra $\pi(A(K' \oplus K'))$ by the formula $\theta'(\pi(a(f \oplus g))) = \pi(a(V'f \oplus W'g))$, $f, g \in K'$, can be extended on the factor $M_{P'}$, $P' = \begin{pmatrix} R' & R'^{1/2}(I-R')^{1/2} \\ R'^{1/2}(I-R')^{1/2} & I-R' \end{pmatrix}$ if and only if $\begin{pmatrix} V' & 0 \\ 0 & W' \end{pmatrix} P' - P' \begin{pmatrix} V' & 0 \\ 0 & W' \end{pmatrix} \in s_2$. The unitary operator $\begin{pmatrix} V' & 0 \\ 0 & W' \end{pmatrix}$ in the space $K' \oplus K'$ satisfies this condition and correctly define θ'. The automorphism θ' has the property $\theta'(\Gamma \otimes \Gamma) = \Gamma \otimes \Gamma$, such that $\theta'(b(f)) = b(W'f)$, $f \in K'$. The factor $M_P \subset M_{P'}$ is invariant under the action of θ'. Considering the restriction we obtain $\theta'|_{M_P} = \theta$. Note that in the case of $V = W$, the endomorphism θ is a regular extension of α in the sense of [8]. $\triangle$

Now let $(V_t)_{t \in R_+}$ be a C_0-semigroup of isometrical operators in K. Then a family of quasifree endomorphisms on M_R defined by the formula $\alpha_t(\pi(a(f \oplus 0))) = \pi(a(V_t f \oplus 0))$, $f \in K$, $t \in R_+$, is a E_0-semigroup. Involve an expanding family of Hilbert spaces $(\mathcal{H}_t)_{t \in R_+}$ embedded in $F(K)$ such that $\mathcal{H}_t$ is generated by all vectors $a^{\#}(f_1)a^{\#}(f_2)\dots a^{\#}(f_n))\Omega$, $f_i \in ker V_t^*$, $1 \le i \le n$, where $a^{\#} = a^*$ or a. The family $(\mathcal{H}_t)_{t \in R_+}$ is a product-system of Hilbert spaces (see [8]). Consider a product-system $K_t = \mathcal{H}_t \otimes J\mathcal{H}_t$, $t \in R_+$. Let $(\theta_t)_{t \in R_+}$ be a regular extension $(\alpha_t)_{t \in R_+}$, then

$$K_t = \theta_t(|\Omega \otimes \Omega >< \Omega \otimes \Omega|)\mathcal{H},\ t \ge 0.$$

By this way, the product-system associated with the regular extension is obtained by the doubling of the product-system associated with the initial semigroup. Note that in the common case (see [22]) $\theta_t(P)$, where P is a one-dimensional projection, is not obliged to be a monotonous increasing family of projections. Such situation can characterize the complete compatability with the exact state (in this case it is

$\omega_R)$.

3. Inner *-automorphisms and the cocycle conjugacy on the hyperfinite factor $\mathcal{M}_R$.

In [9] it was proved that a quasifree derivation δ of $\mathcal{A}(\mathcal{K})$ acting on the generating elements by the formula $\delta(a(f)) = a(df),\ f \in dom\ d$, where d is some scewhermitian operator, is inner iff $d \in s_1$. Then the automorphism obtained by the quasifree lifting of an unitary $e^d, d \in s_1$, is inner. Notice that $e^d - I \in s_1$. On the other side every inner automorphism α can not be represented in the form $\alpha = e^\delta$, where δ is some inner derivation. In the following theorem we give the necessary and sufficient condition of an innerness of $B_R(W)$ in the terms of $W,\ WR = RW$.

Theorem 2. *The quasifree automorphism $B_R(W)$ of the hyperfinite factor $\mathcal{M}_R$ is inner iff $R^{1/2}(I-R)^{1/2}(W-I) \in s_2$.*

Remark. *The results of [18] yelds a sufficiency and a necessity of the condition of theorem 2 in the case of a pure point spectrum of W. Thus we need to prove a necessity of one.*

Proof of theorem 2 (necessity).

The quasifree automorphism $B_R(W)$ has a form $B_R(W)(\cdot) = \mathcal{U} \cdot \mathcal{U}^*,\ \mathcal{U} \in \mathcal{M}_R$ by the condition. In the appendix we show that the generating elements of commutant $\mathcal{M}'_R$ are $\mathbf{1}, b(f) = \mathit{\Gamma} \otimes \mathit{\Gamma}\pi(a(\mathit{0} \oplus f)),\ b^*(f) = \pi(a^*(\mathit{0} \oplus f))\mathit{\Gamma} \otimes \mathit{\Gamma},\ f \in \mathcal{K}$. Thus $\mathcal{U}\pi(a(\mathit{0} \oplus f))\mathcal{U}^* = \mathcal{U}\mathit{\Gamma} \otimes \mathit{\Gamma}\mathcal{U}^*\mathit{\Gamma} \otimes \mathit{\Gamma}\pi(a(\mathit{0} \oplus f)),\ f \in \mathcal{K}$. Let us show that $\mathcal{U}\Gamma \otimes \Gamma\mathcal{U}^*\Gamma \otimes \Gamma = 1$. Note that $(\Gamma \otimes \Gamma)^2 = I,\ (\Gamma \otimes \Gamma)^* = \Gamma \otimes \Gamma$ therefore $\mathcal{U}\Gamma \otimes \Gamma\mathcal{U}^*\Gamma \otimes \Gamma$ equals $\mathbf{1}$ or $-\mathbf{1}$. By this way $\mathcal{U}\pi(a(0 \oplus f))\mathcal{U}^* = \pi(a(0 \oplus f))$ or $-\pi(a(0 \oplus f)),\ f \in \mathcal{K}$. In the first case the quasifree automorphism $B(W \oplus I)$ is unitary implementable, in the second case the quasifree automorphism $B(W \oplus (-I))$ is that. Therefore by theorem 1, $R^{1/2}(I-R)^{1/2}(W-I) \in s_2$ or $R^{1/2}(I-R)^{1/2}(W+I) \in s_2$. In the first case the theorem is proved. Suppose $R^{1/2}(I-R)^{1/2}(W+I) \in s_2$. Then $R^{1/2}(I-R)^{1/2}(-W-I) \in s_2$ and the automorpism $B_R(-W)$ is inner. Thus the automorphism $B_R(-I) = B_R(-W)B_R(W)$ is inner. It is a contradiction by [18]. Thus we proved $\mathcal{U}\pi(a(0 \oplus f))\mathcal{U}^* = \pi(a(0 \oplus f))$. Therefore the automorphism $B(W \oplus I)$ of the C^*-algebra $\pi(\mathcal{A}(\mathcal{K} \oplus \mathcal{K}))$ obtained by the lifting of $W \oplus I$ is unitary implementable and $R^{1/2}(I-R)^{1/2}(W-I) \in s_2$ by theorem 1. $\triangle$

Let $(U_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ be C_0-semigroups of unitaries in a Hilbert space $\mathcal{K}$ commuting with the operator R.

Theorem 3. *The quasifree semigroups* $(B_R(U_t))_{t\geq 0}$ *and* $(B_R(V_t))_{t\geq 0}$ *on the hyperfinite factor* $\mathcal{M}_R$ *are cocycle conjugate iff* $R^{1/2}(I-R)^{1/2}(U_t - V_t) \in s_2,\ t \geq 0$.

Proof of theorem 3 is based on theorem 1, theorem 2 and one result of [20] that we formulate in the following lemma:

Lemma. *Let σ-week continuous groups of *-automorphisms* $\alpha = (\alpha_t)_{t\in\mathbf{R}}$ *and* $\beta = (\beta_t)_{t\in\mathbf{R}}$ *on the* W^**-factor* $\mathcal{M}$ *having separable predual* $\mathcal{M}_*$ *be such that* **-automorphisms* $\beta_{-t}\alpha_t$ *are inner for all* $t \in \mathbf{R}$*. Then the group* α *and* β *are cocycle conjugate.*

Proof of theorem 3.

Necessity. Let the semigroups $(B_R(U_t))_{t\geq 0}$ and $(B_R(V_t))_{t\geq 0}$ be cocycle conjugate. Then $B_R(U_t)(\cdot) = W_t B_R(V_t)(\cdot) W_t^*,\ W_t \in \mathcal{M}_R,\ t \geq 0$. Fix $t \geq 0$. The automorphism $B(V_t \oplus V_t)$ is unitary implementable by theorem 1. Therefore the automorphism $B(U_t \oplus V_t)(\cdot) = W_t B(V_t \oplus V_t)(\cdot) W_t^*$ is unitary implementable too. The result follows from theorem 1.

Sufficiency. The result follows from theorem 2 and the lemma. $\triangle$

4. The cocycle conjugacy of quasifree endomorphisms semigroups.

Let $U = (U_t)_{t\geq 0}$ and $V = (V_t)_{t\geq 0}$ be C_0-semigroups of isometries in a Hilbert space $\mathcal{K}$ commuting with the operator R. Then there are C_0-semigroups of unitaries $U' = (U'_t)_{t\geq 0}$ and $V' = (V'_t)_{t\geq 0}$ in a Hilbert space $\mathcal{K}',\ \mathcal{K} \subset \mathcal{K}'$, being minimal unitary dilations of semigroups U and V and the positive contraction R' commuting with U', V' and satisfying the relation $U'_t R = R' U_t,\ V'_t R = R' V_t,\ t \geq 0$.

Definition. *Two C_0-semigroups of isometries U and V are called to be approximating each other if* $U'_t - V'_t \in s_2$ *and* $U'_t V'^*_t|_{\mathcal{K}'\ominus\mathcal{K}} = I,\ t \geq 0$.

Theorem 4. *Let a C_0-semigroup of isometries U approximates a C_0-semigroup of isometries V. Then the quasifree semigroup* $(B_R(U_t))_{t\geq 0}$ *is cocycle conjugate to the quasifree semigroup* $(B_R(V_t))_{t\geq 0}$.

Proof of theorem 4.

Theorem 3 leads to an existence of a cocycle $(\mathcal{W}_t)_{t\geq 0}$ such that $B_{R'}(U'_t)(\cdot) = \mathcal{W}_t B_{R'}(V'_t)(\cdot)\mathcal{W}_t^*,\ \mathcal{W}_t \in \mathcal{M}_{R'}(\mathcal{K}'),\ t \geq 0$. We show that the condition $U'_t V'^*_t|_{\mathcal{K}'\ominus\mathcal{K}} = I$ implies $\mathcal{W}_t \in \mathcal{M}_R(\mathcal{K})$. Note that if this condition holds a family of unitaries $W_t = U'_t V'^*_t|_{\mathcal{K}},\ W_t - I \in s_2,\ t \geq 0$, is correctly defined. The quasifree lifting of $(W_t)_{t\geq 0}$ determines a family of inner automorphisms $B_R(W_t)(x) = \mathcal{W}'_t x \mathcal{W}'^*_t,\ \mathcal{W}'_t, x \in$

$\mathcal{M}_R(\mathcal{K}),\ t \geq 0$. We constructed $(\mathcal{W}'_t)_{t\geq 0}$ such that $\mathcal{W}'_t x \mathcal{W}'^*_t = \mathcal{W}_t x \mathcal{W}^*_t,\ x \in \mathcal{M}_{R'}(\mathcal{K}'),\ t \geq 0$. Therefore $\mathcal{W}_t = e^{ic(t)}\mathcal{W}'_t,\ c(t) \in \mathbf{R},\ t \geq 0$, and $\mathcal{W}_t \in \mathcal{M}_R(\mathcal{K})$.

5. Continuous semigroups of isometries in a Hilbert space.

It is useful to remind that an isometry V in a Hilbert space $\mathcal{K}$ is called completely nonunitary if there is no subspace $\mathcal{K}_0 \subset \mathcal{K}$ reducing V to an unitary. Every C_0-semigroup of completely nonunitary isometries $(V_t)_{t\geq 0}$ is unitary equivalent to its model that is a C_0-semigroup of shifts $(S_t)_{t\geq 0}$ acting in a Hilbert space $\mathcal{K}' = H \otimes L_2(0,+\infty)$ by the formula $(S_t f)(x) = f(x-t)$ for $x > t,\ (S_t f)(x) = 0$ for $0 < x < t,\ f \in \mathcal{K}'$. Here H is some Hilbert space of the dimension n equal to the deficiency index of the generator of $(V_t)_{t\geq 0}$ (we call n by the deficiency index of $(V_t)_{t\geq 0}$ in the following). Let $(V_t)_{t\geq 0}$ be a C_0-semigroup of isometries in a Hilbert space $\mathcal{K}$ and $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ be the Wold decomposition of $\mathcal{K}$, where $\mathcal{K}_0$ reduces $(V_t)_{t\geq 0}$ to a C_0-semigroup of unitaries and $\mathcal{K}_1$ reduces $(V_t)_{t\geq 0}$ to a C_0-semigroup of completely nonunitary isometries. We shall say $(V_t)_{t\geq 0}$ satisfies the condition N if the C_0-semigroup of unitaries $(V_t|_{\mathcal{K}_0})_{t\geq 0}$ is uniformly continuous that is $||V_t|_{\mathcal{K}_0} - V_s|_{\mathcal{K}_0}|| \to 0,\ t \to s,\ s,t \geq 0$.

Theorem 5 ([1-3,5-6]). *Let a C_0-semigroup of isometries $(V_t)_{t\geq 0}$ in a Hilbert space $\mathcal{K}$ with a deficiency index $n > 0$ satisfy the condition N.*

Then there is a C_0-semigroup of completely nonunitary isometries $(S_t)_{t\geq 0}$ with a deficiency index n approximating $(V_t)_{t\geq 0}$ in the sence of the definition of part 4.

In the proof of the theorem we use the complex analysis in the Hardy space (see [21]). Let $(\lambda_k)_{1\leq k\leq N},\ N \leq +\infty$, be a system of complex numbers satisfying the following properties,

$$Re\lambda_k < 0,\ |Im\lambda_k| < R,\ 1 \leq k \leq N,\ \sum_{k=1}^{N} |Re\lambda_k| < +\infty, \tag{1}$$

where R is some positive number. In this case the formula $B(\lambda) = \prod\limits_{k=1}^{N} \frac{\lambda+\overline{\lambda}_k}{\lambda-\lambda_k},\ \lambda \in \mathbf{C}$, defines an analitic function being regular in the semiplane $Re\lambda > 0$ and equaling the unit by module on the imaginary axis. The function $B(\lambda)$ is called the Blaschke product.

Proposition 1. *Let the numbers $(\lambda_k)_{1\leq k\leq N}$ satisfy the condition (1). Then the*

Blaschke product $B(\lambda)$ constructed of $(\lambda_k)_{1\le k\le N}$ can be estimated as follows:

$$|B(\lambda)|<C_1,\ |\lambda|>C_2,\ B(\lambda)=1-\frac{C_3}{\lambda}+o(\frac{1}{\lambda}),\ |\lambda|\to+\infty,\ \lambda\in\mathbf{C},$$

where C_1, C_2 and C_3 are some positive constants.

Proof.

$$lnB(\lambda)=\sum_{k=1}^{N} ln\frac{1+\frac{\overline{\lambda_k}}{\lambda}}{1-\frac{\lambda_k}{\lambda}}=-\frac{2s}{\lambda}+o(\frac{1}{\lambda}),\ |\lambda|\to+\infty,$$

where $s=-\sum_{k=1}^{N} Re\lambda_k$, $0<s<+\infty$ by the condition (1). $\triangle$

Let us consider a Hilbert space $\mathcal{K}=L_2(0,+\infty)$. Let $P_{[t_1,t_2]}$ designate a projection on a subspace of $\mathcal{K}$ consisting of functions $f(x)=0$ for $0<x<t_1,\ t_2<x<+\infty$. Let us define an isometry Θ acting in $\mathcal{K}$ by the formula $\Theta=\mathcal{F}^{-1}B\mathcal{F}$, where $\mathcal{F}$ and B are the Fourier transformation and an operator of the multiplication by the Blaschke product correspondently.

Proposition 2. *Let the conditions of proposition 1 be hold.*
Then $\Delta_{t,\delta}=P_{[t,t+\delta]}\Theta P_{[t,t+\delta]}-P_{[t,t+\delta]}\in s_2,\ 0<t,\delta<+\infty,\ ||\Delta_{t,\delta}||_2=O(\delta^{1/2}),\ \delta\to 0.$

Proof.

Fix $t\ge 0$. Let $\mu_{k,\delta}=-\frac{1}{2|k|}+i\frac{2\pi k}{\delta},\ k\in\mathbf{Z},\ \delta>0$. Let us consider a family of functions $f_{k,\delta}(x)=\frac{(-2Re\mu_{k,\delta})^{1/2}}{(e^{2Re\mu_{k,\delta}t}-e^{2Re\mu_{k,\delta}(t+\delta)})^{1/2}}e^{\mu_{k,\delta}x},\ t<x<t+\delta,\ f_{k,\delta}(x)=0,\ 0<x<t,\ t+\delta<x<+\infty,\ k\in\mathbf{Z}$. The family $(f_{k,\delta})_{k\in\mathbf{Z}}$ is the Riesz basis of a Hilbert space $H=P_{[t,t+\delta]}\mathcal{K}$ (see [21]). So there is a bounded operator V having bounded revers in $\mathcal{K}$ such that the family $(Vf_{k,\delta})_{k\in\mathbf{Z}}$ is an orthogonal basis of H. Therefore to prove proposition 2 it is sufficient to prove a convergence of the series $\sum_{k\in\mathbf{Z}}||\Delta_{t,\delta}f_{k,\delta}||^2$ and to investigate its dependence on δ. Let $f_{k,\delta}=\frac{(-2Re\mu_{k,\delta})^{1/2}}{e^{2Re\mu_{k,\delta}t}-e^{2Re\mu_{k,\delta}(t+\delta))^{1/2}}}e^{\mu_{k,\delta}x},\ f^{(2)}_{k,\delta}=P_{[t+\delta,+\infty]}f^{(1)}_{k,\delta},\ t<x<+\infty,\ f^{(1)}_{k,\delta}=f^{(2)}_{k,\delta}=0,\ 0<x<t,\ k\in\mathbf{Z}$. Then $f_{k,\delta}=f^{(1)}_{k,\delta}-f^{(2)}_{k,\delta},\ k\in\mathbf{Z}$ and $||\Delta_{t,\delta}f_{k,\delta}||^2\le 2(||\Delta_{t,\delta}f^{(1)}_{k,\delta}||^2+||\Delta_{t,\delta}f^{(2)}_{k,\delta}||^2),\ k\in\mathbf{Z}$. By this way,

$$||\Delta_{t,\delta}f^{(i)}_{k,\delta}||^2=2(||f^{(i)}_{k,\delta}-Re(\Theta f^{(i)}_{k,\delta},f^{(i)}_{k,\delta})),\ k\in\mathbf{Z},\ i=1,2,$$

$$||f^{(1)}_{k,\delta}||^2=\frac{e^{Re\mu_{k,\delta}t}}{(e^{Re\mu_{k,\delta}t}-e^{Re\mu_{k,\delta}(t+\delta)})}=-\frac{1}{2Re\mu_{k,\delta}\delta}+o(1),\ |k|\to+\infty,$$

$$||f_{k,\delta}^{(2)}||^2 = \frac{e^{Re\mu_{k,\delta}(t+\delta)}}{(e^{Re\mu_{k,\delta}t} - e^{Re\mu_{k,\delta}(t+\delta)})} = -\frac{1}{2Re\mu_{k,\delta}\delta} + o(1),\ |k| \to +\infty.$$

Using the Laplace transformation technics one can obtain

$$(\Theta f_{k,\delta}^{(i)}, f_{k,\delta}^{(i)}) = B(\mu_{k,\delta})||f_{k,\delta}^{(i)}||^2,\ i = \overline{1,2}.$$

It follows from proposition 1 that

$$||\Delta_{t,\delta} f_{k,\delta}^{(i)}||^2 = ||f_{k,\delta}^{(i)}||^2(1 - ReB(\mu_{k,\delta})) = \frac{1}{2\delta|\mu_{k,\delta}|^2} + o(\frac{1}{2\delta|\mu_{k,\delta}|^2}) =$$

$$\frac{\delta}{8\pi^2 k^2} + o(\frac{\delta}{k^2}),\ |k| \to +\infty,\ \delta \to 0,\ i = \overline{1,2},$$

and

$$||\Delta_{t,\delta}||_2^2 \le C\sum_{k\in\mathbf{Z}} ||\Delta_{t,\delta} f_{k,\delta}||^2 \le 2C\sum_{k\in\mathbf{Z}} (||\Delta_{t,\delta} f_{k,\delta}^{(1)}||^2 + ||\Delta_{t,\delta} f_{k,\delta}^{(2)}||^2) = O(\delta),\ \delta \to 0,$$

where C is some positive constant, that implies $||\Delta_{t,\delta}||_2 = O(\delta^{1/2}),\ \delta \to 0.$ $\triangle$

Proof of theorem 5.

Let the operator d be a generator of uniformly continuous semigroup of unitaries $(U_t)_{t\ge 0}$ being unitary part of some semigroup of isometries. In accordance with the Von Neumann theorem, given a scewhermitian operator d there is a bounded scewhermitian operator $D \in \sigma_2$ such that the scewhermitian operator $d + D$ has a purely point spectrum. So the operator $d + D$ is a generator of some uniformly continuous semigroup of unitaries $(V_t)_{t\ge 0}$ having a purely point spectrum. The semigroups $(U_t)_{t\ge 0}$ and $(V_t)_{t\ge 0}$ are known to be bonded by the relation $V_t - U_t = \int\limits_0^t U_{t-s} D V_s ds,\ t \ge 0$, therefore $V_t - U_t \in \sigma_2,\ t \ge 0$ and $||V_{t+\delta} - U_{t+\delta} - V_t + U_t||_2 = O(\delta),\ \delta \to 0,\ t \ge 0.$

For the semigroup $(V_t)_{t\ge 0}$ is uniformly continuous its spectrum lies in a circle of radius $R > 0$ in the complex plane. Let $(i\mu_k)_{1\le k\le N}$ be eigenvalues of the generator of $(V_t)_{t\ge 0}$ numbering in order of decreasing of its modules. By this way, $|\mu_k| < R,\ 1 \le k \le N \le +\infty$. Let complex numbers $(\lambda_k)_{1\le k\le N}$ be such that $Im\lambda_k = -\mu_k,\ 1 \le k \le N$, and the real parts collected in the accordance of the condition (1). The condition (1) allows to define the Blaschke product associated with $(\lambda_k)_{1\le k\le N}$ (see above).

In what follows we suppose the deficiency index to be 1. It will prove the theorem because every C_0-semigroup of isometries having nonzero deficiency index decomposes in an orthogonal sum of a C_0-semigroup of isometries having a deficiency index 1 and, probably, a C_0-semigroup of completely nonunitary isometries. We shall prove the existence of a C_0-semigroup of isometries in a Hilbert space $\mathcal{K} = L_2(0,+\infty)$ that is unitary equivalent to given a semigroup of isometries with a deficiency index 1 and a purely point spectrum of its unitary part consisting of numbers $(i\mu_k)_{1\le k\le N}$.

Let $(S_t)_{t\ge 0}$ be a C_0-semigroup of shifts in a Hilbert space $\mathcal{K} = L_2(0,+\infty)$. Let us consider a family of functions $f_n(x) = (-2Re\lambda_n)^{1/2}e^{\lambda_n x},\ 1 \le n \le N$. The condition (1) implies an uncomleteness of the system $(f_n)_{1\le n\le N}$ in $\mathcal{K}$ (the condition of the convergence of the Blaschke product). Thus a subspace $\mathcal{K}_1$ being a linear envelope of $(f_n)_{1\le n\le N}$ does not coincide with $\mathcal{K}$ and defines a subspace $\mathcal{K}_0$ being invariant under an action of $(S_t)_{t\ge 0}$ and completely describing by the condition of an orthogonality to all functions f_n such that $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ and the isometry $\Theta : \mathcal{K} \to \mathcal{K},\ \Theta = \mathcal{F}^{-1}B\mathcal{F},\ \Theta S_t = S_t\Theta,\ t \ge 0$, where $\mathcal{F}$ and B are the Fourier transformation and an operator multiplying by the Blaschke product, defines $\mathcal{K}_0$ and $\mathcal{K}_1$ by the formula $\mathcal{K}_0 = \Theta\mathcal{K}$. The semigroup $(S_t)_{t\ge 0}$ is intertwined by the operator Θ with its restriction on the subspace $\mathcal{K}_0$: $S_t|_{\mathcal{K}_0}\Theta = \Theta S_t,\ t \ge 0$. The isometric operator $\Theta :\ \mathcal{K} \to \mathcal{K}$ sets an unitary map $\mathcal{K} \to Ran\Theta = \mathcal{K}_0$. Hence the semigroups $(S_t)_{t\ge 0}$ and $(S_t|_{h_0})_{t\ge 0}$ are unitarily equivalent such that the deficiency index of the semigroup $(S_t|_{\mathcal{K}_0})_{t\ge 0}$ consisting of completely nonunitary isometries in $\mathcal{K}_0$ coincides with the deficiency index of the semigroup $(S_t)_{t\ge 0}$ and equals 1.

Let the system of functions $(g_n)_{n\in\mathbf{N}}$ be obtained by a succesive orthogonalization of the system $(f_n)_{n\in\mathbf{N}}$. Let us determine a C_0-semigroup $(V_t)_{t\ge 0}$ of isometries in $\mathcal{K}$ as follows

$$V_t|_{\mathcal{K}_0} = S_t|_{\mathcal{K}_0},\ V_t g_n = e^{iIm\lambda_n t}g_n,\ t \ge 0, n \in \mathbf{N}. \tag{2}$$

We shall show that for isometries $V_t,\ t \ge 0$, describing in (2) the following conditions hold, $V_tS_t^* - P_{[t,+\infty]} \in \sigma_2, ||V_t - S_t||_2 \le ||V_tS_t^* - P_{[t,+\infty)}||_2 = O(t^{1/2}),\ t \to 0$.

Fix $t > 0$. We need to prove a convergence of the series $\sum\limits_{n=1}^{+\infty} ||(V_tS_t^* - P_{[t,+\infty)})f_n||^2$ for some orthogonal basis $(f_n)_{n\in\mathbf{N}}$ of $\mathcal{K}$. Choose for this purpose an arbitrary addition of the system $(S_tg_n)_{n\in\mathbf{N}}$ up to an orthogonal basis of $\mathcal{K}$.

Notice that $V_tS_t^* - P_{[t,+\infty)} = P_{[0,t]}V_tS_t^* + (P_{[t,+\infty)}V_tS_t^* - P_{[t,+\infty)})$ and $(V_tS_t^* -$

$P_{[t,+\infty)})|_{\mathcal{K}_t} = 0$, where $\mathcal{K}_t$ designates an orthogonal addition of a linear envelope of vectors $(S_t g_n)_{n\in\mathbf{N}}$. An element $S_t^* g_n$ belongs to a linear envelope of elements $g_i,\ i = 1, n$, such that $(S_t^* g_n, g_n) = e^{\lambda_n t},\ 1 \le n \le N$. By this way,

$$||(P_{[t,+\infty)}V_tS_t^* - P_{[t,+\infty)})S_tg_n||^2 = 1 + ||P_{[t,+\infty)}g_n||^2 - 2Re(V_tg_n, S_tg_n) <$$
$$2(1 - Re(V_tg_n, S_tg_n)) = 2(1 - e^{-Re\lambda_n}),\ n \in \mathbf{N}. \tag{3}$$

For the operator $P_{[0,t]}V_tS_t^*$ we have the estimate using the Bessel inequality:

$$||P_{[0,t]}V_tS_t^*S_tg_n||^2 = ||P_{[0,t]}g_n||^2 = 1 - (P_{[t,+\infty)}g_n, g_n) < 1 - |(S_t^*g_n, g_n)|^2$$
$$= 1 - e^{-2Re\lambda_n t},\ n \in \mathbf{N}. \tag{4}$$

It follows from (3), (4) and (1) that $\sum\limits_{i=1}^{+\infty} ||(V_tS_t^* - P_{[t,+\infty)})f_i||^2 = \sum\limits_{n=1}^{+\infty}(||(P_{[t,+\infty)}V_tS_t^* - P_{[t,+\infty)})S_tg_n||^2 + ||(P_{[0,t]}V_tS_t^*S_tg_n||^2) = \sum\limits_{n=1}^{+\infty}(-4Re\lambda_n t + o(Re\lambda_n)) = O(t),\ t \to 0$. Hence $||V_tS_t^* - P_{[t,+\infty)}||_2 = O(t^{1/2}),\ t \to 0$.

Notice that $V_t - S_t = V_t|_{\mathcal{K}_1} - P_{\mathcal{K}_1}S_t|_{\mathcal{K}_1},\ t \ge 0$, and the C_0-semigroup $(V_t|_{\mathcal{K}_1})_{t\ge 0}$ is uniformly continuous by the condition. The condition $V_t - S_t \in s_2,\ t \ge 0,\ ||V_t - S_t||_2 = O(t^{1/2}),\ t \to 0$, and the uniform continuity of $(V_t|_{\mathcal{K}_1})_{t\ge 0}$ implies that the family $(V_t - S_t)_{t\ge 0}$ is continuous in $||\cdot||_2$. In fact, $||(V_{t+\delta} - S_{t+\delta} - V_t + S_t||_2 = ||P_{\mathcal{K}_1}(V_{t+\delta} - S_{t+\delta} - V_t + S_t)P_{\mathcal{K}_1}||_2 \le ||P_{\mathcal{K}_1}(V_\delta - I)(V_t - S_t)P_{\mathcal{K}_1}||_2 + ||(V_\delta - S_\delta)V_t||_2 + ||(V_t - S_t)(V_\delta - S_\delta)||_2 \le ||P_{\mathcal{K}_1}(V_\delta - I)P_{\mathcal{K}_1}||\,||V_t - S_t||_2 + ||V_\delta - S_\delta||_2(1 + ||V_t - S_t||_2) \to 0,\ \delta \to 0$.

Now we get for operators $\Delta_t = V_t - S_t,\ t \ge 0$ the following estimations:

$$\Delta_t \in s_2,\ ||\Delta_{t+\delta} - \Delta_t||_2 \to 0,\ ||\Delta_\delta||_2 = O(\delta^{1/2}),\ \delta \to 0,\ t \ge 0. \tag{5}$$

To complete the proof we need to show that there are C_0-semigroups $(V_t')_{t\ge 0}$ and $(S_t')_{t\ge 0}$ being unitary dilations of $(V_t)_{t\ge 0}$ and $(S_t)_{t\ge 0}$ which satisfy the conditions $V_t' - S_t' \in s_2,\ V_t'S_t'^*|_{\mathcal{K}'\ominus\mathcal{K}} = I,\ t \ge 0$. Let us define unitary dilations in a Hilbert space $\mathcal{K}' = \mathcal{K} \oplus \mathcal{K}$ by the formula

$$S_t'(f \oplus g)(x) = ((S_tf)(x) + (P_{[0,t]}g)(t - x)) \oplus (S_t^*g)(x),$$

$$V_t'(f \oplus g)(x) = ((V_tf)(x) + (\Theta(P_{[0,t]}g)(t - \cdot))(x)) \oplus (S_t^*g)(x),$$

$$x, t \geq 0,\ \ f, g \in \mathcal{K}.$$

Now the result follows from (5) and proposition 2. $\triangle$

6. The class of the cocycle conjugacy of the quasifree flow of shifts on the hyperfinite factor $\mathcal{M}_\nu$.

Let $V = (V_t)_{t\geq 0}$ be a C_0-semigroup of isometries having uniformly continuous unitary part (see part 5). Then theorem 5 implies the existence of a C_0-semigroup of completely nonunitary isometries $S = (S_t)_{t\geq 0}$ in $\mathcal{K}$ approximating V in the sence of part 4. Semigroups V and S have same deficiency indeces. If $(S_t)_{t\geq 0}$ is a C_0-semigroup of completely nonunitary isometries, then the quasifree semigroup $(B_\nu(S_t))_{t\geq 0}$ consists of shifts. It follows from theorem 4 that the following assertion holds.

Theorem 6 ([2-5]). *Let a C_0-semigroup of isometries $(V_t)_{t\geq 0}$ with the deficiency index $n > 0$ have uniformly continuous unitary part.*

Then there is a C_0-semigroup of completely nonunitary isometries $(S_t)_{t\geq 0}$ with the deficiency index n such that the quasifree semigroup $(B_\nu(V_t))_{t\geq 0}$ is cocycle conjugate to the flow of shifts $(B_\nu(S_t))_{t\geq 0}$.

Theorem 6 describes the class of the cocycle conjugacy of the quasifree flow of shifts on the hyperfinite factor $\mathcal{M}_\nu$ so one can reformulate it as follows,

Theorem 6'. *Let $V = (V_t)_{t\geq}$ be a C_0-semigroup of isometries in a Hilbert space $\mathcal{K}$ having uniformly continuous unitary part and $S = (S_t)_{t\geq 0}$ be a C_0-semigroup of completely nonunitary isometries in $\mathcal{K}$ with the same deficiency index as in V.*

Then there is an unitary U in $\mathcal{K}$ such that the quasifree semigroup $(B_\nu(UV_tU^*))_{t\geq 0}$ is cocycle conjugate to the flow of shifts $(B_\nu(S_t))_{t\geq 0}$.

Notice that there is an analogue of the result of theorem 6 for discrete quasifree semigroups, see [5,7].

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Appendix. The commutant of $\mathcal{M}_R$.

Note that the operators

$$b(f) = \Gamma \otimes \Gamma\pi(a(0 \oplus f)), b^*(f) = \Gamma \otimes \Gamma\pi(a^*(0 \oplus f)),\ \ f \in \mathcal{K},$$

belong to a commutant of the hyperfinite factor $\mathcal{M}_R$. Therefore its generate the W^*-algebra $\mathcal{N} \subset \mathcal{M}'_R$. The formula $Sx\Omega \otimes \Omega = x^*\Omega \otimes \Omega,\ x \in \mathcal{M}_\nu$ correctly defines an antilinear operator S in $\mathcal{H}$ (see [4]). Let $S = \mathcal{J}\Delta^{1/2}$ be a polar decomposition of S. Then an antiisometrical part $\mathcal{J}$ of the operator S is antiunitary and is called a modular involution of $\mathcal{M}_R$. The linear (unbounded) positive operator Δ is called a modular operator (see [14]). Simple calculation gives the following formula for $\mathcal{J}$,

$$\mathcal{J} f_1 \Lambda \ldots \Lambda f_n \otimes g_1 \Lambda \ldots \Lambda g_m = J g_m \Lambda \ldots \Lambda J g_1 \otimes J f_n \Lambda \ldots \Lambda J f_1,\ \ \mathcal{J}\Omega \otimes \Omega = \Omega \otimes \Omega.$$

By this way,

$$\mathcal{J}\pi(a(f \oplus 0))\mathcal{J} = b^*(f), \quad f \in \mathcal{K},$$

therefore $\mathcal{J}\mathcal{M}_{\mathcal{R}}\mathcal{J} = \mathcal{M}'_{\mathcal{R}}$ and $\mathcal{N} = \mathcal{M}'_R$.

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